Oberseminar Algebra, Zahlentheorie
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Towards the computation of minimal models of symplectic quotient singularities

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Symplectic quotient singularities

Mori dream spaces and Cox rings

An algorithm for $\mathcal{R}(X)$

Namikawa's hyperplanes

Linear quotients

V: symplectic $\mathbb{C}$-vector space, $\dim(V) = n < \infty$

$G \leq \mathrm{Sp}(V)$: finite group

The *linear quotient* of V by G is the affine variety

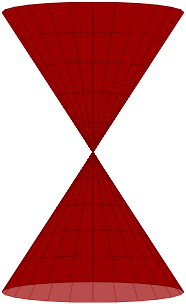
$$V/G = \operatorname{Spec} \mathbb{C}[V]^G \text{ “=” space of orbits.}$$

Example

$V = \mathbb{C}^2$ and $G = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \leq \mathrm{GL}_2(\mathbb{C})$

$$\mathbb{C}[V]^G \cong \mathbb{C}[x, y, z]/\langle xy - z^2 \rangle$$

$$V/G = V(xy - z^2) \subseteq \mathbb{A}^3$$



Fact: V/G is singular.

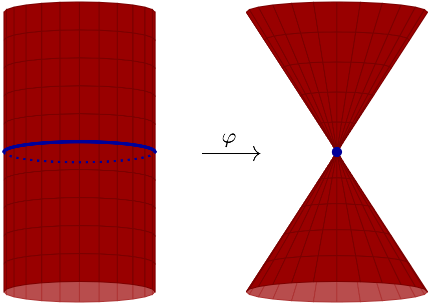
Symplectic varieties

The variety V/G inherits the symplectic structure of V.
$\Longrightarrow V/G$ is a symplectic variety (Beauville, 2000).

A symplectic resolution of V/G is a projective resolution $\varphi : X \to V/G$ with X a symplectic variety and φ an isomorphism of symplectic varieties over the smooth locus.

Example

Minimal resolutions of Kleinian singularities



Existence of symplectic resolutions

In general, there is no symplectic resolution of V/G (for example, $G = \langle -I_4 \rangle$).

Classification of quotients admitting a symplectic resolution: ongoing work since ∼2000.

Only 45 groups left to classify.

Symplectic resolutions only exist in special cases.

Fact: There is always a symplectic partial resolution – a $\mathbb{Q}$-factorial terminalization – of V/G (Birkar–Cascini–Hacon–McKernan, 2010).

Question: Can we construct $X \to V/G$ algorithmically?

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Mori dream spaces and Cox rings

In the following: $\varphi : X \to V/G$ symplectic resolution (or a $\mathbb{Q}$-factorial terminalization)

Proposition (Namikawa, 2015)

The variety X is a relative Mori dream space over V/G.

Equivalently: The Cox ring $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra.

Assume that $\mathrm{Cl}(X)$ is free. The Cox ring of X is the algebra

$$\mathcal{R}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Example

$\mathcal{R}(\mathbb{P}^n) = \mathbb{C}[x_0, \ldots, x_n]$ with the standard grading.

GIT quotients (for algebraists)

The class group $\mathrm{Cl}(X)$ is a finitely generated abelian group.
In all 'interesting cases', the group $\mathrm{Cl}(X)$ is free (S., 2024).

Given a $D \in \mathrm{Div}(X)$, we obtain a positively graded algebra

$$S(D) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O}_X(kD))$$

and the variety $X(D) = \operatorname{Proj} S(D)$.

For some D (ample), we have $X(D) \cong X$.

All $\mathbb{Q}$-factorial terminalizations of V/G arise in this way.

A conceptual algorithm

How to construct X:

(1) Compute $\mathcal{R}(X)$ (without knowing X!)

(2) Find a good $D \in \mathrm{Div}(X)$

(3) Compute $S(D)$

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An embedding

Proposition (Grab, 2019)

There is an injective graded morphism

$$\Theta : \mathcal{R}(X) \to \mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}].$$

Fact: We have $\mathrm{Cl}(V/G) \cong \mathrm{Hom}(G, \mathbb{C}^\times)(= \Delta)$ (Benson, 1993).

Theorem (Arzhantsev–Gaĭfullin, 2010)

There is a graded isomorphism

$$\mathcal{R}(V/G) \cong \mathbb{C}[V]^{[G,G]},$$

where the graded component of $\chi \in \Delta$ is given by

$$\mathbb{C}[V]^{[G,G]}_\chi = \{f \in \mathbb{C}[V]^{[G,G]} \mid \gamma.f = \chi(\gamma)f \text{ for all } \gamma \in G\}.$$

Example

Let $G \leq \mathrm{GL}_4(\mathbb{C})$ acting on $V = \mathbb{C}^4$ be generated by

$$r = \begin{pmatrix} \zeta_3 & . & . & . \\ . & \zeta_3^{-1} & . & . \\ . & . & \zeta_3^{-1} & . \\ . & . & . & \zeta_3 \end{pmatrix} \text{ and } s = \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix}.$$

Then $[G, G] = \langle r \rangle \cong C_3$, so $\mathrm{Cl}(V/G) \cong \Delta \cong \mathrm{Ab}(G) = \mathbb{Z}/2\mathbb{Z}$.

The ring $\mathcal{R}(V/G) \cong \mathbb{C}[V]^{[G,G]}$ is generated by:

In degree χ_0:

$$x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ x_1^3 + x_2^3,\ x_3^3 + x_4^3,\ x_2x_3^2 + x_1x_4^2,\ x_2^2x_3 + x_1^2x_4$$

In degree χ_1:

$$x_1x_3 - x_2x_4,\ x_1^3 - x_2^3,\ x_3^3 - x_4^3,\ x_2x_3^2 - x_1x_4^2,\ x_2^2x_3 - x_1^2x_4$$

Reduction to $\mathcal{R}(V/G)$

There is an injective graded morphism

$$\Theta : \mathcal{R}(X) \to \mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}].$$

Let $\mathrm{Cl}(X)^{\mathrm{free}} \cong \mathbb{Z}^m$, so $\mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.

There are valuations $v_i : \mathbb{C}[V] \setminus \{0\} \to \mathbb{Z}$ coming from certain elements of G (Ito–Reid, 1996).

Given $f \in \mathcal{R}(V/G)$, we have

$$f \otimes \prod_{i=1}^{m} t_i^{v_i(f)} \in \mathrm{im}(\Theta).$$

Aim: Find homogeneous $f_1, \ldots, f_k \in \mathcal{R}(V/G)$ that give rise to generators of $\mathrm{im}(\Theta)$.

Homogeneous Khovanskii bases

Mise en place I

Assumption: $m = 1$, there is just one valuation $v = v_1$.

Idea: Find special generators whose 'initial terms' generate the 'initial algebra'.

We have a filtration

$$F_{v \geq a} = \{f \in \mathbb{C}[V] \setminus \{0\} \mid v(f) \geq a\} \cup \{0\}$$

with $a \in \mathbb{Z}$ and the associated graded algebra

$$\mathrm{gr}_v(\mathbb{C}[V]) = \bigoplus_{a \in \mathbb{Z}} F_{v \geq a} / F_{v > a}.$$

Homogeneous Khovanskii bases

Mise en place II

Let $f \in \mathbb{C}[V] \setminus \{0\}$ with valuation $v(f) = a$.

We write $\mathrm{in}_v(f) \in \mathrm{gr}_v(\mathbb{C}[V])$ for the residue class of f in $F_{v \geq a}/F_{v > a}$.

Let $f \in \mathbb{C}[V]^{[G,G]} \setminus \{0\}$ be Δ-homogeneous of degree $\deg_\Delta(f) = \chi$.

We define

$$\mathrm{in}_v^\Delta(f) = \mathrm{in}_v(f) \otimes \chi \in \mathrm{gr}_v(\mathbb{C}[V]) \otimes_\mathbb{C} \mathbb{C}\Delta.$$

Let $\mathrm{in}_v^\Delta(\mathbb{C}[V]^{[G,G]})$ be the algebra generated by

$$\{\mathrm{in}_v^\Delta(f) \mid f \in \mathbb{C}[V]^{[G,G]} \ \Delta\text{-homogeneous}\}.$$

Example

With G as before, we have $\mathrm{gr}_v(\mathbb{C}[V]) = \mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_4]$, but with a non-standard grading.

Let $y_1 = \frac{1}{2}(x_1 + x_2)$, $y_2 = \frac{1}{2}(x_1 - x_2)$, $y_3 = \frac{1}{2}(x_3 + x_4)$ and $y_4 = \frac{1}{2}(x_3 - x_4)$.

The ring $\mathrm{gr}_v(\mathbb{C}[V])$ is generated in degree 0 and 1 via

$$\mathrm{gr}_v(\mathbb{C}[V])_0 = \langle y_1, y_3 \rangle_{\mathbb{C}} \text{ and } \mathrm{gr}_v(\mathbb{C}[V])_1 = \langle y_2, y_4 \rangle_{\mathbb{C}}.$$

For example:

$$x_1 = y_1 + y_2 \Longrightarrow \mathrm{in}_v(x_1) = y_1$$

$$x_1 x_2 = (y_1 + y_2)(y_1 - y_2) \Longrightarrow \mathrm{in}_v^{\Delta}(x_1 x_2) = y_1^2 \otimes \chi_0$$

Homogeneous Khovanskii bases

Let $\mathcal{B} \subseteq \mathbb{C}[V]^{[G,G]}$ be a set of Δ-homogeneous generators of $\mathbb{C}[V]^{[G,G]}$ as a $\mathbb{C}$-algebra.

We call $\mathcal{B}$ a Δ-homogeneous Khovanskii basis of $\mathbb{C}[V]^{[G,G]}$ with respect to v, if $\{\mathrm{in}_v^\Delta(f) \mid f \in \mathcal{B}\}$ generates $\mathrm{in}_v^\Delta(\mathbb{C}[V]^{[G,G]})$.

Theorem (Yamagishi, 2018; Grab, 2019; S., 2024+)

Homogeneous elements $f_1, \dots, f_k \in \mathbb{C}[V]^{[G,G]}$ give rise to generators of $\mathrm{im}(\Theta)$ if and only if $\{f_1, \dots, f_k\}$ is a Δ-homogeneous Khovanskii basis of $\mathbb{C}[V]^{[G,G]}$ with respect to v.

Computing a Khovanskii basis

(1) Let $\mathcal{B} = \{f_1, \ldots, f_k\}$ be Δ-homogeneous generators of $\mathbb{C}[V]^{[G,G]}$.

(2) Compute the kernel $\langle h_1, \ldots, h_l \rangle$ of the morphism

$$\mathbb{C}[X_1, \ldots, X_k] \to \mathrm{in}_v^\Delta(\mathbb{C}[V]^{[G,G]}),\ X_i \mapsto \mathrm{in}_v^\Delta(f_i).$$

(3) 'Reduce' $h_i(f_1, \ldots, f_k)$ with respect to $\mathcal{B}$ and add non-zero remainders to $\mathcal{B}$.

(4) If elements were added: go to (2), else: done.

Example

In the example, the given generators of $\mathbb{C}[V]^{[G,G]}$ already form a homogeneous Khovanskii basis.

Generators of $\mathcal{R}(X) \subseteq \mathbb{C}[V] \otimes \mathbb{C}[t^{\pm 1}]$ are given by

$$x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ x_1^3 + x_2^3,\ x_3^3 + x_4^3,\ x_2x_3^2 + x_1x_4^2,\ x_2^2x_3 + x_1^2x_4$$
$$(x_1x_3 - x_2x_4)t,\ (x_1^3 - x_2^3)t,\ (x_3^3 - x_4^3)t,\ (x_2x_3^2 - x_1x_4^2)t,\ (x_2^2x_3 - x_1^2x_4)t, t^{-2}$$

where the grading by $\mathrm{Cl}(X) = \mathbb{Z}$ is via the degree of t.

Some remarks

A finite Khovanskii basis may not exists in a more general setting, but here it does ($\rightarrow$ MDS).

The reduction algorithm may not terminate in a more general setting, but here it does.

Generalization to $m > 1$ valuations exists (MUVAK bases).

There is an algorithm, but no filtration and no reduction algorithm available.

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Namikawa's hyperplanes

There is a wall-and-chamber structure in a subcone of $\mathbb{R}^m$.

Chambers $\hat{=}$ isomorphism classes of $\mathbb{Q}$-factorial terminalizations.

Walls give divisors D such that $X(D)$ is not a $\mathbb{Q}$-factorial terminalization.

Namikawa, 2015: The walls come from a hyperplane arrangement and there is a reflection group acting on this arrangement.

This requires that V/G is symplectic.