Retreat of the SFB-195

On the computation of Cox rings of minimal models of symplectic linear quotients

... `using Oscar`

Johannes Schmitt
TU Kaiserslautern
25th March 2022

Symplectic linear quotients

Cox rings

McKay correspondence

Yamagishi's algorithm

Symplectic Groups

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Example

For $V = \mathbb{C}^{2n}$ and $\omega(v, w) := v^\top J_n w$ with $J_n := \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, we have

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid g^\top J_n g = J_n\} \ .$$

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Let $\mathfrak{h}$ be a vector space. Then $\mathfrak{h} \oplus \mathfrak{h}^*$ is symplectic via

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Fact: $\mathrm{Sp}(V) \leq \mathrm{SL}(V)$.

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Let $\mathbf{C}_2 := \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \leq \mathrm{Sp}_2(\mathbb{C})$. Then

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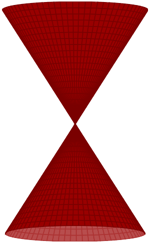

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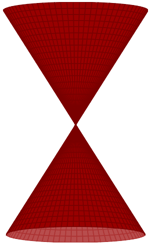

Classical fact

The variety V/G is smooth if and only if G is generated by reflections, i.e. $g \in \mathrm{GL}(V)$ with $\mathrm{rk}(g - 1) = 1$.

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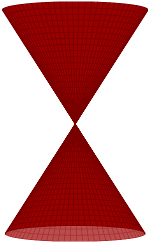

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Corollary

If V is symplectic and $G \leq \mathrm{Sp}(V)$, then V/G is singular.

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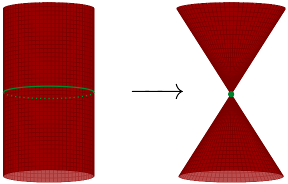

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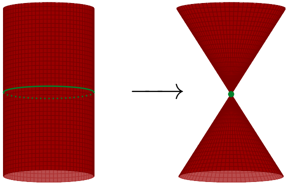

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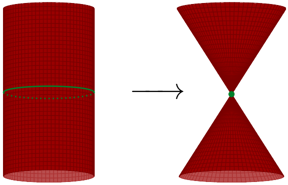

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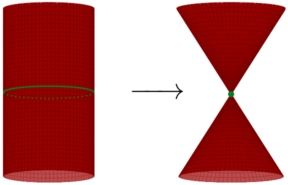

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If V/G admits a symplectic resolution, then G is generated by symplectic reflections, i.e. $g \in G$ with $\mathrm{rk}(g - 1) = 2$.

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Example

Let $W \leq \mathrm{GL}(\mathfrak{h})$ be a complex reflection group. Then $W \leq \mathrm{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$ is a symplectic reflection group.

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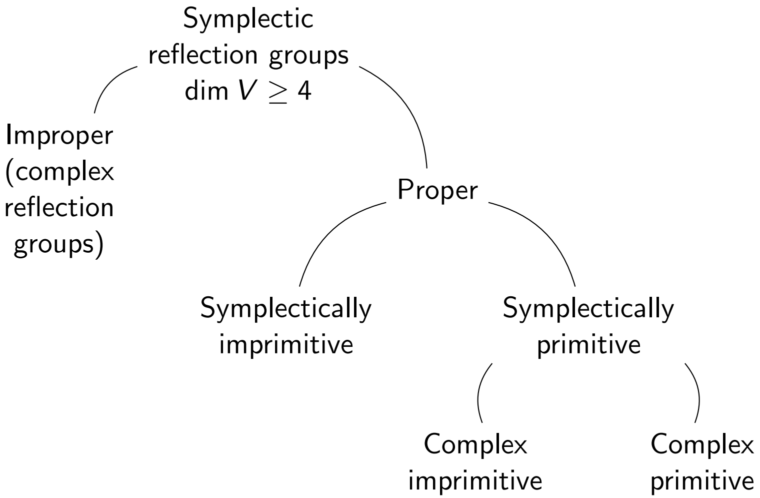

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- Symplectic reflection groups $\dim V \geq 4$
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# open cases	Improper	Symplectically imprimitive	Complex imprimitive	Complex primitive
	DONE!	DONE!	39	6

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2. Recover all minimal models via variation of GIT quotient ($\rightarrow$ project A23).

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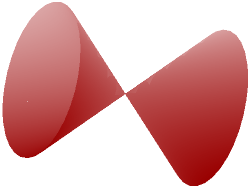

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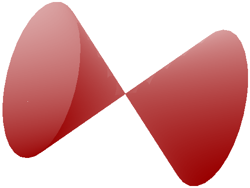

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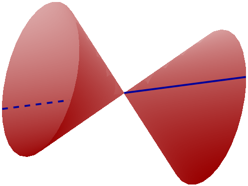

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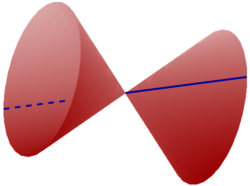

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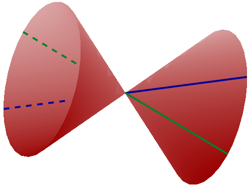

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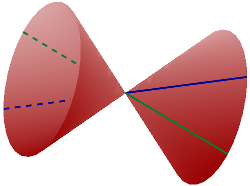

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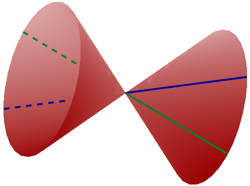

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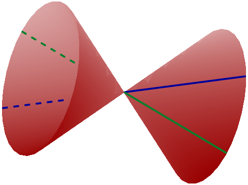

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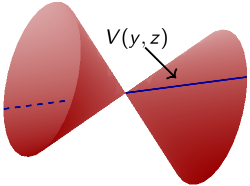

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Example

Let $Y = \mathbb{P}^n$, $D = V(x_0)$ und $H = \mathbb{Z}D$. For $k \in \mathbb{Z}$ we have

$$\Gamma(Y, \mathcal{O}_Y(kD)) \cong \{f \in \mathbb{C}[\underline{x}] \mid f \text{ homogeneous}, \ \deg f = k\} \cup \{0\} \, .$$

Hence $\mathcal{R}(\mathbb{P}^n) \cong \mathbb{C}[\underline{x}]$ with the standard grading.

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Theorem (Arzhantsev–Gaĭfullin, 2010)

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Example

For $G = \mathbf{C}_2$ the group $[G, G]$ is trivial, so $\mathcal{R}(\mathbb{C}^2/G) \cong \mathbb{C}[x_1, x_2]$ graded by $\mathbf{C}_2$.

Remember the goal:

Compute the Cox ring $\mathcal{R}(X)$ of any minimal model X of V/G without prior knowledge of X.

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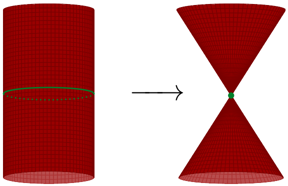
Example

The element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathbf{C}_2$ is the single symplectic reflection of $\mathbf{C}_2$.
Accordingly, the minimal resolution of $\mathbb{C}^2/\mathbf{C}_2$ has one exceptional prime divisor.



Valuations

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Let $g \in G$ be a symplectic reflection of order r and write

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For $0 \neq h = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \lambda_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]$, where $n = \dim V$, we define

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Valuations

Let $g \in G$ be a symplectic reflection of order r and write

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Theorem (Ito–Reid, 1996)

Let $E \in \operatorname{Div}(X)$ be the exceptional divisor corresponding to g. Then the valuation on $\mathbb{C}(X) = \mathbb{C}(V)^G$ corresponding to E is given by $\frac{1}{r}\nu_g$.

Symplectic linear quotients

Cox rings

McKay correspondence

Yamagishi's algorithm

Embedding the Cox ring

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There is a surjective morphism of graded rings $\mathcal{R}(X) \to \mathcal{R}(V/G)$ induced by φ via the isomorphism $\varphi^* : \mathbb{C}(V/G) \to \mathbb{C}(X)$ and the push-forward $\varphi_* : \mathrm{Cl}(X) \to \mathrm{Cl}(V/G)$.

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Theorem (Grab, Yamagishi, 2018+)

Homogeneous generators $f_1, \ldots, f_s \in \mathcal{R}(V/G)$ give rise to generators of $\operatorname{im} \Theta$ if and only if they satisfy $(*f)$ for every homogeneous $f \in \mathcal{R}(V/G)$.

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Proposition (Yamagishi, 2018)

We have $I \subseteq J$. The generators $f_1, \dots, f_s$ satisfy $(*f)$ for every homogeneous $f \in \mathcal{R}(V/G)$ if and only if $I = J$.

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- matrix groups over arbitrary number fields