Symplectic Resolutions

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Symplectic Groups

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For $n = 1$: $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$. In general: $\mathrm{Sp}_{2n}(\mathbb{C}) \leq \mathrm{SL}_{2n}(\mathbb{C})$.

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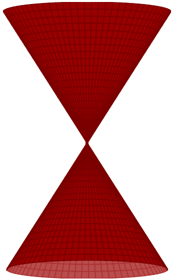

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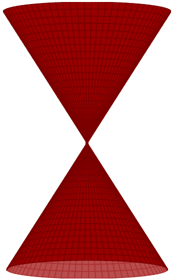
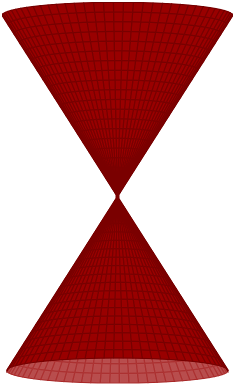


Hilbert–Noether

The ring $\mathbb{C}[V]^G$ is an affine $\mathbb{C}$-algebra.

Resolutions



V/G

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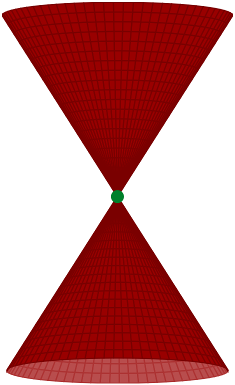

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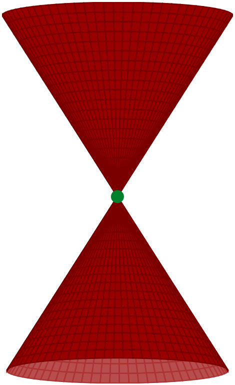
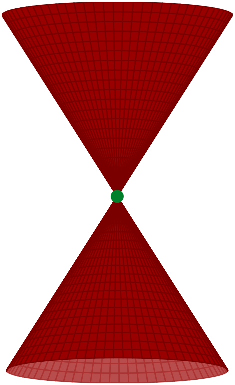


V/G

Shephard–Todd, Chevalley

The variety V/G is smooth if and only if G is generated by reflections, i.e. $g \in \mathrm{GL}(V)$ with $\mathrm{rk}(g-1) = 1$.

Resolutions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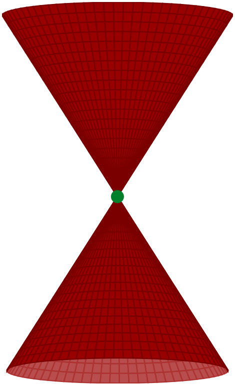

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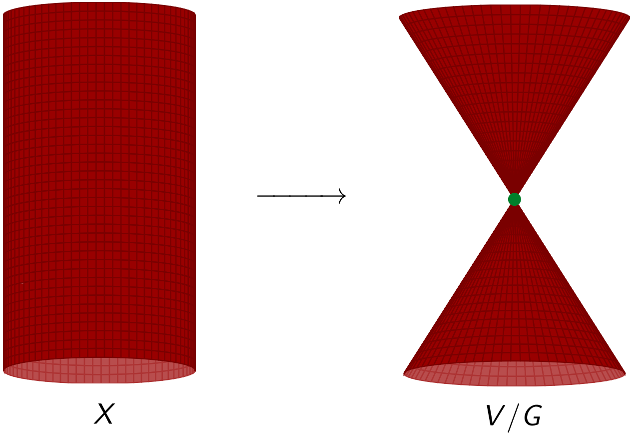

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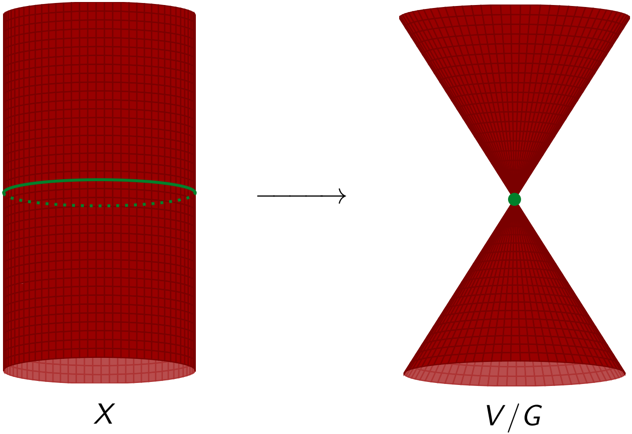

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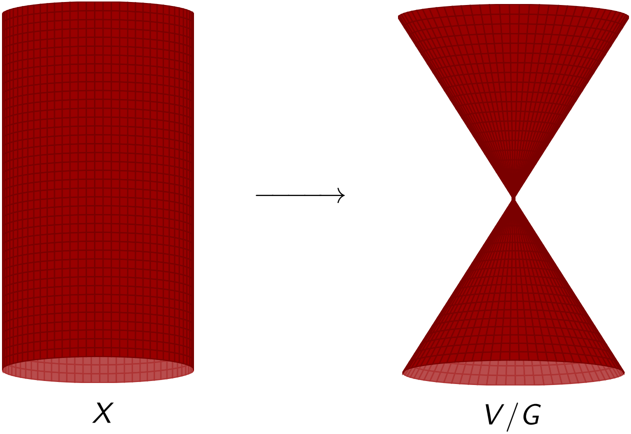

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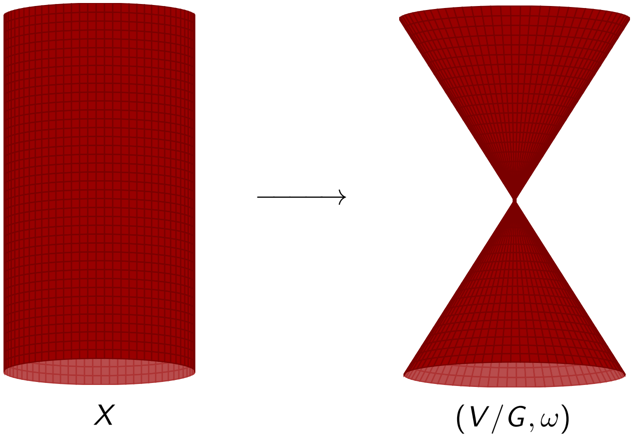

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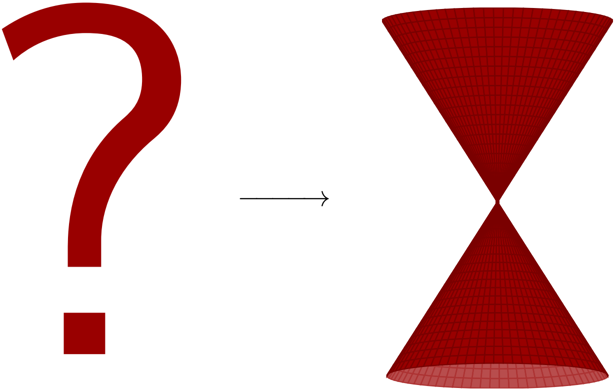

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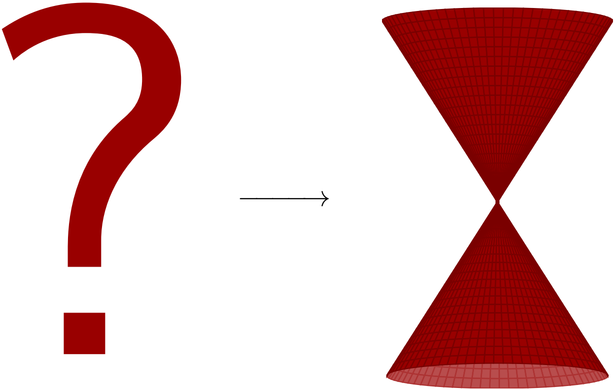

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In general no such resolution exists!

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We only need to consider irreducible tuples (V, ω, G).

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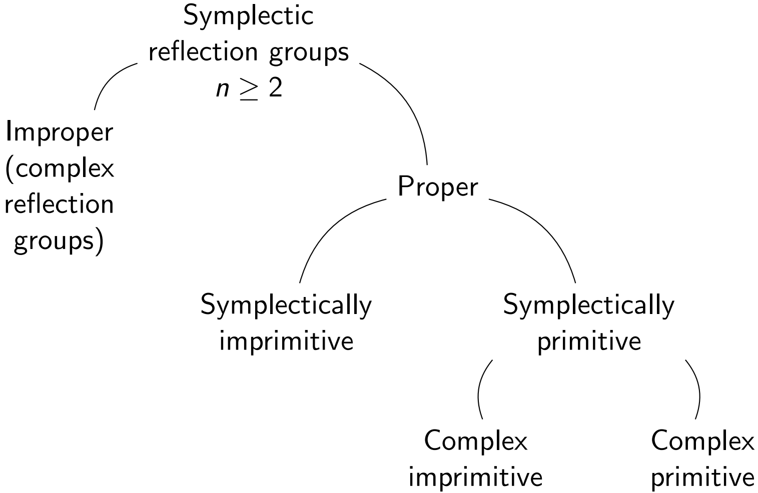
Verbitsky

If V/G admits a symplectic resolution, then G is generated by symplectic reflections, i.e. $g \in G$ with $\mathrm{rk}(g - 1) = 2$.

What is known?



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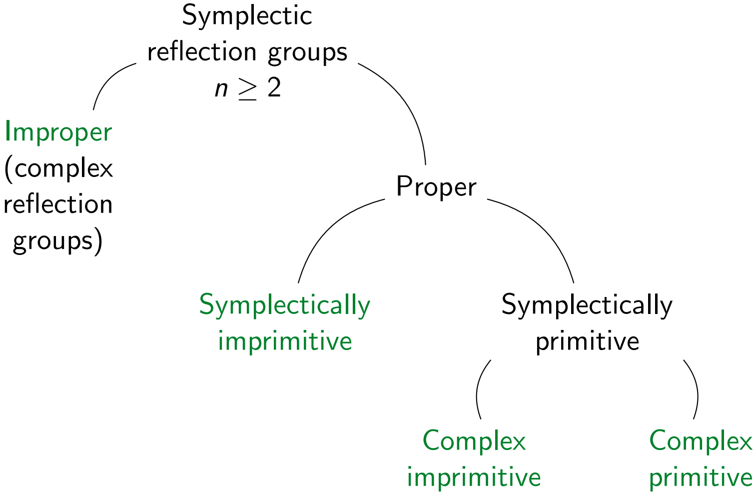
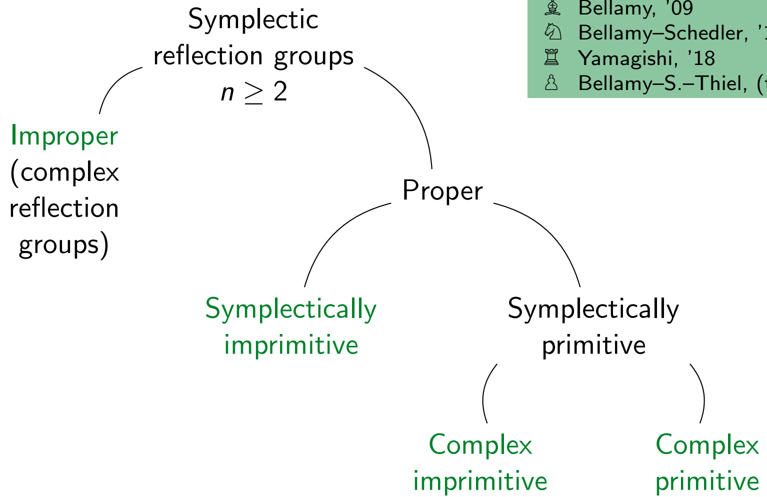


What is known?

- ♔ Etingof–Ginzburg, '02
- ♕ Gordon, '03
- ♗ Bellamy, '09
- ♘ Bellamy–Schedler, '16
- ♖ Yamagishi, '18
- ♙ Bellamy–S.–Thiel, (to appear)



	♔, ♕, ♗	♘, ♖	♙	♘, ♙
Groups with resolution	$G(m, 1, n)$, G_4	$K \wr S_n$ ($K \leq \mathrm{SL}_2(\mathbb{C})$), $Q_8 \times_{\mathbb{Z}/2\mathbb{Z}} D_8$	none (so far)	none (so far)
# open cases	DONE!	DONE!	39	9