Towards the Classification of Symplectic Linear Quotient Singularities Admitting a Symplectic Resolution

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Joint work with
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The Classification Problem

The Groups in Question

Mise en Place

Construction of the Module

Symplectic Groups

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Example

For $V = \mathbb{C}^{2n}$ and $\omega(v, w) := v^\top J_n w$ with $J_n := \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, we have

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid g^\top J_n g = J_n\} .$$

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Let $\mathfrak{h}$ be a vector space. Then $\mathfrak{h} \oplus \mathfrak{h}^*$ is symplectic via

$$\omega\big((v, f), (w, g)\big) = g(v) - f(w) \ .$$

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Fact: $\mathrm{Sp}(V) \leq \mathrm{SL}(V)$.

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Let $\mathbf{C}_2 := \langle \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \rangle \leq \mathrm{Sp}_2(\mathbb{C})$. Then

$$\mathbb{C}[u, v, w]/\langle uv - w^2 \rangle \cong \mathbb{C}[x, y]^{\mathbf{C}_2} .$$

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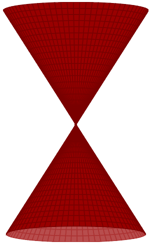

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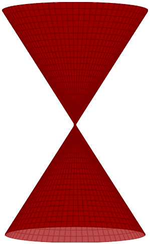

Classical fact

The variety V/G is smooth if and only if G is generated by reflections, i.e. $g \in \mathrm{GL}(V)$ with $\mathrm{rk}(g - 1) = 1$.

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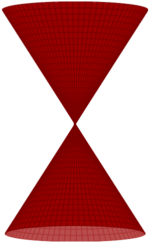

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Corollary

If V is symplectic and $G \leq \mathrm{Sp}(V)$, then V/G is singular.

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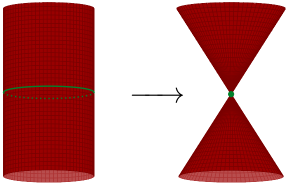

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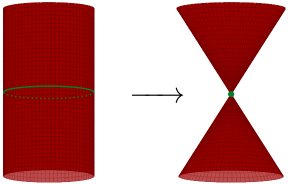

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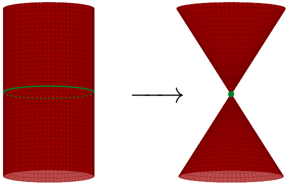

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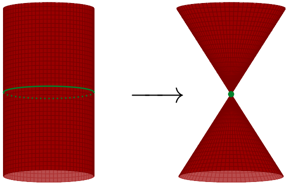

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In general, those do not exist!

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If V/G admits a symplectic resolution, then G is generated by symplectic reflections, i.e. $g \in G$ with $\mathrm{rk}(g-1) = 2$.

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Example

Let $W \leq \mathrm{GL}(\mathfrak{h})$ be a complex reflection group. Then $W^\vee \leq \mathrm{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$ is a symplectic reflection group.

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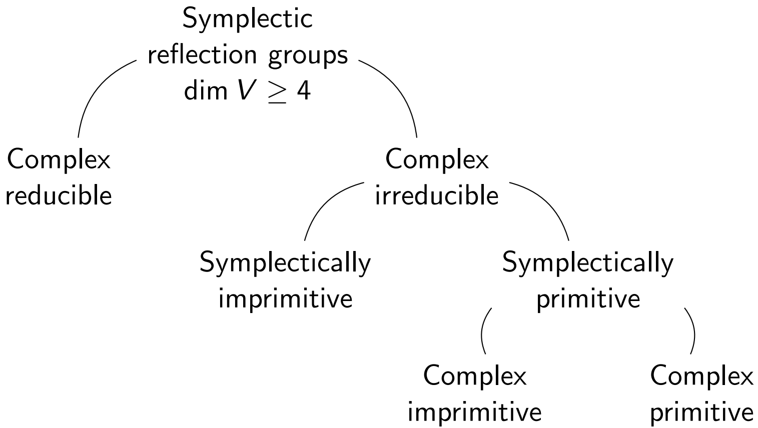

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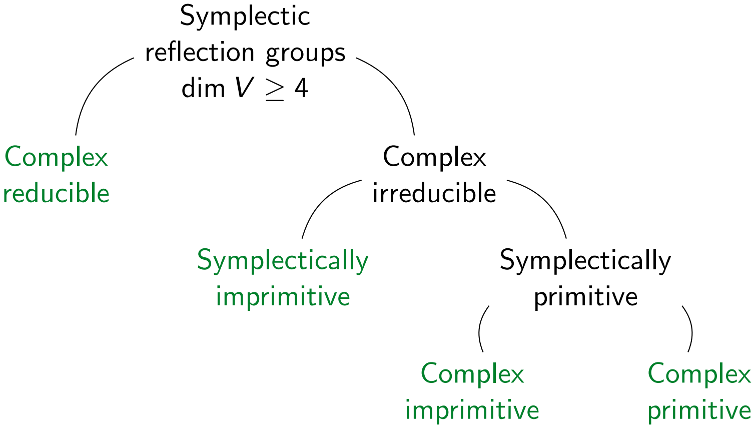

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    - Symplectically imprimitive
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Using Champ (Thiel, 2013), we arrived at 39 open cases.

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- complex imprimitive, so there exists such a decomposition into (not necessarily symplectic) subspaces.

For these groups we have $\dim V = 4$ (Cohen, 1980), so we may assume $V = \mathbb{C}^4$.

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We have the following infinite families of groups in $\mathrm{GL}_2(\mathbb{C})$:

(1) $\mu_d\mathrm{T}$, with d a multiple of 6,

(2) $\mu_d\mathrm{O}$, with d a multiple of 4,

(3) $\mu_d\mathrm{I}$, with d a multiple of 4, 6, or 10,

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The group OT_d is defined as follows. We have $\mathrm{T} \trianglelefteq \mathrm{O}$ with $\mathrm{O}/\mathrm{T} \cong C_2$, so $\mathrm{O} = \langle \mathrm{T}, g \rangle$. We then set

$$\mathrm{OT}_d = \bigcup_{\substack{k=0 \\ k \text{ even}}}^{2d-2} \zeta_{2d}^k \mathrm{T} \cup \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{2d-1} \zeta_{2d}^k g \mathrm{T} \ .$$

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Theorem (Cohen, 1980)

The four infinite families arising in this way give all the symplectically primitive complex imprimitive symplectic reflection groups up to conjugacy.

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μ_{12}O	15	μ_{24}O	11
μ_4I	22	μ_6I	20
μ_{10}I	16	μ_{12}I	21
μ_{20}I	17	μ_{30}I	18
μ_{60}I	19		
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Note: This splits the four families in 17 $(2 + 4 + 7 + 4)$ infinite families.
E.g. μ_{6d}T for d odd (with $H_0 = G_5$) and μ_{6d}T for d even (with $H_0 = G_7$).

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Problem: We don't know a presentation of $\mathbb{C}[V]^G$ in general. Also, $\mathbb{C}[V]^G$ does not "know" about the group action.

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Problem: We don't know a presentation of $\mathbb{C}[V]^G$ in general. Also, $\mathbb{C}[V]^G$ does not "know" about the group action.

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Etingof–Ginzburg, 2002

Such deformations are given by the symplectic reflection algebras $\mathsf{H}_{\mathbf{c}}(V, G)$.

Symplectic Reflection Algebras

Write $\mathcal{S}(G) \subseteq G$ for the set of symplectic reflections.

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Let $\mathsf{H}_{\mathbf{c}}(G, V)$ be the quotient of $T(V) \rtimes G$ by the ideal

$$\left\langle [u, v] - \sum_{g \in \mathcal{S}(G)} \mathbf{c}(g) \omega_g(u, v) g \;\middle|\; u, v \in V \right\rangle .$$

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From now on: $\mathrm{H}_{\mathbf{c}}(G) := \mathrm{H}_{\mathbf{c}}(G, V)$.

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The same strategy was already used for the complex reducible symplectic reflection groups.

Reducing to Groups I

Rigid Representations

Let M by a G-module. This extends to a module $\tilde{M}$ over $T(V) \rtimes G$ by letting V act trivially.

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Technical detail

The module M is $\mathbf{c}$-rigid if and only if

$$\sum_{g \in \mathcal{S}(G)} \mathbf{c}(g)\omega_g(u, v)\rho_M(g) = 0$$

for all $u, v \in V$, where ρ_M is the representation corresponding to M.

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Example (Thiel, 2014)

Let $D_d \leq \mathrm{GL}_2(\mathbb{C})$ be a dihedral group of order ≥ 10 and consider $D_d^\vee \leq \mathrm{Sp}_4(\mathbb{C})$. Then almost all simple D_d-modules are **c**-rigid for all **c**.

Reducing to Groups II

Baby Verma Modules

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For any irreducible W-module λ there exists an irreducible $\mathsf{H}_{\mathbf{c}}(W^\vee)$-module $L(\lambda)$ of dimension $\dim L(\lambda) \leq |W|$.

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One can explicitly compute these modules using an algorithm by Thiel (2015).

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This means we can "split" $\mathbf{c}$ into two parameters $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$ with

$$\mathbf{c}_1 : \mathcal{S}(H_0^{\vee}) \to \mathbb{C} \text{ and } \mathbf{c}_2 : \mathcal{S}(D_d) \to \mathbb{C} \,.$$

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This gives (sub)algebras

$$\mathsf{H}_{\mathbf{c}_1}(H_0^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(H^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}_2}(D_d) \,.$$

Construction of the Module

$$\mathsf{H}_{\mathbf{c}_1}(H_0^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(H^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}_2}(D_d)$$

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For $\lambda \in \mathrm{Irr}(H)$, we have $\lambda|_{H_0} \in \mathrm{Irr}(H_0)$ giving rise to a simple $\mathsf{H}_{\mathbf{c}_1}(H_0^\vee)$-module $L(\lambda|_{H_0})$ with $\dim L(\lambda|_{H_0}) \leq |H_0|$.

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The module $L(\lambda|_{H_0})$ is a simple $\mathsf{H}_{\mathbf{c}_1}(H^\vee)$-module as well.

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We can induce any simple $\mathsf{H}_{\mathbf{c}_1}(H^\vee)$-module L to an $\mathsf{H}_{\mathbf{c}_1}(E(H))$-module M with $\dim M = 2 \dim L$.

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Theorem

This is an $\mathsf{H}_{\mathbf{c}}(E(H))$-module if and only if all the constituents of $M|_{D_d}$ are $\mathbf{c}_2$-rigid w.r.t. $\mathsf{H}_{\mathbf{c}_2}(D_d)$.

$$\mathsf{H}_{\mathbf{c}_1}(H_0^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(H^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}_2}(D_d)$$

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We can construct an $\mathsf{H}_{\mathbf{c}}(E(H))$-module M with $\dim M < |E(H)|$, if we can find $\lambda \in \mathrm{Irr}(H)$ such that $L(\lambda|_{H_0})|_{D_d}$ is $\mathbf{c}_2$-rigid for arbitrary $\mathbf{c}_2$ and $H_0 \lneq H$.

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Using theoretical bounds on d we obtain:

Theorem

Such a module exists except in possibly 73 cases.

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H_0	Groups containing H_0 as largest reflection group	H_0	Groups containing H_0 as largest reflection group
$\mu_6\mathrm{T}$	$\mu_d\mathrm{T},\ d \in \{6, 18, 30\}$	$\mu_{12}\mathrm{T}$	$\mu_d\mathrm{T},\ d \in \{12, 24, 36, 48\}$
$\mu_4\mathrm{O}$	$\mu_d\mathrm{O},\ d \in \{4, 20, 28\}$	$\mu_8\mathrm{O}$	$\mu_d\mathrm{O},\ d \in \{8, 16, 32, 40, 56, 64\}$
$\mu_{12}\mathrm{O}$	$\mu_d\mathrm{O},\ d \in \{12, 36, 60\}$	$\mu_{24}\mathrm{O}$	$\mu_d\mathrm{O},\ d \in \{24, 48, 72, 96\}$
$\mu_4\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{4, 8, 16, 28, 32, 44, 52, 56, 64\}$	$\mu_6\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{6, 18, 42, 54, 66, 78\}$
$\mu_{10}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{10, 50, 70\}$	$\mu_{12}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{12, 24, 36, 48, 72, 84,$
$\mu_{20}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{20, 40, 80, 100, 140\}$		$96, 108, 132, 144\}$
$\mu_{30}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{30, 90, 150\}$	$\mu_{60}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{60, 120, 180, 240\}$
OT_2	$\mathrm{OT}_d,\ d \in \{2, 10, 14\}$	OT_4	$\mathrm{OT}_d,\ d \in \{4, 20\}$
OT_6	$\mathrm{OT}_d,\ d \in \{6, 18, 30\}$	OT_{12}	$\mathrm{OT}_d,\ d \in \{12, 36\}$

Conclusion (Refinement)

We can construct an $\mathsf{H}_{\mathbf{c}}(E(H))$-module M with $\dim M < |E(H)|$, if we can find $\lambda \in \mathrm{Irr}(H)$ such that $L(\lambda|_{H_0})|_{D_d}$ is $\mathbf{c}_2$-rigid for arbitrary $\mathbf{c}_2$ and $H_0 \lneq H$.

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Using the MAGMA-package CHAMP (Thiel, 2013) we can improve the theoretical bounds and obtain sharp bounds in many cases:

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Refined Theorem

Such a module exists except in possibly 39 cases. In at least 18 of them no such module exists.

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μ_{30}I	μ_dI, $d \in \{30, 90, 150\}$	μ_{60}I	μ_dI, $d \in \{60, 120, 180, 240\}$
OT$_2$	OT$_d$, $d \in \{2, 10, 14\}$	OT$_4$	OT$_d$, $d \in \{4, 20\}$
OT$_6$	OT$_d$, $d \in \{6, 18, 30\}$	OT$_{12}$	OT$_d$, $d \in \{12, 36\}$

Conclusion (Refinement)

We can construct an $\mathrm{H}_{\mathbf{c}}(E(H))$-module M with $\dim M < |E(H)|$, if we can find $\lambda \in \mathrm{Irr}(H)$ such that $L(\lambda|_{H_0})|_{D_d}$ is $\mathbf{c}_2$-rigid for arbitrary $\mathbf{c}_2$ and $H_0 \lneq H$.

Using the MAGMA-package CHAMP (Thiel, 2013) we can improve the theoretical bounds and obtain sharp bounds in many cases:

Refined Theorem

Such a module exists except in possibly 39 cases. In at least 18 of them no such module exists.

H_0	Groups containing H_0 as largest reflection group	H_0	Groups containing H_0 as largest reflection group
$\mu_6\mathrm{T}$	$\mu_6\mathrm{T}$	$\mu_{12}\mathrm{T}$	$\mu_{12}\mathrm{T}$
$\mu_4\mathrm{O}$	$\mu_4\mathrm{O}$	$\mu_8\mathrm{O}$	$\mu_d\mathrm{O},\ d \in \{8, 16\}$
$\mu_{12}\mathrm{O}$	$\mu_{12}\mathrm{O}$	$\mu_{24}\mathrm{O}$	$\mu_d\mathrm{O},\ d \in \{24, 48, 72, 96\}$
$\mu_4\mathrm{I}$	$\mu_4\mathrm{I}$	$\mu_6\mathrm{I}$	$\mu_6\mathrm{I}$
$\mu_{10}\mathrm{I}$	$\mu_{10}\mathrm{I}$	$\mu_{12}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{12, 24, 36, 48, 72, 84, 96, 108, 132, 144\}$
$\mu_{20}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{20, 40, 80, 100, 140\}$		
$\mu_{30}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{30, 90, 150\}$	$\mu_{60}\mathrm{I}$	$\mu_d\mathrm{I},\ d \in \{60, 120, 180, 240\}$
OT_2	OT_2	OT_4	OT_4
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$\mu_4\mathrm{I}$	$\mu_4\mathrm{I}$	$\mu_6\mathrm{I}$	$\mu_6\mathrm{I}$
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OT_2	OT_2	OT_4	OT_4
OT_6	OT_6	OT_{12}	OT_{12}

Final result (so far)

Let $G \leq \mathrm{Sp}(V)$ be a symplectically primitive complex imprimitive symplectic reflection group. Then the corresponding quotient V/G does not admit a symplectic resolution except in possibly 39 cases.

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Generalize all this to subgroups of $\mathrm{SL}(V)$.