Retreat of the SFB-TRR 191

Towards the Classification of Symplectic Linear Quotient Singularities Admitting a Symplectic Resolution

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TU Kaiserslautern
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Joint work with
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The Classification Problem

The Groups in Question

Mise en Place

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Symplectic Groups

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Example

For $V = \mathbb{C}^{2n}$ and ω induced by $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ we have

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Let $\mathfrak{h}$ be a vector space. Then $\mathfrak{h} \oplus \mathfrak{h}^* = T^*\mathfrak{h}$ is symplectic via

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Notation: Write $W^\vee$ for this action.

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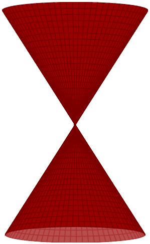

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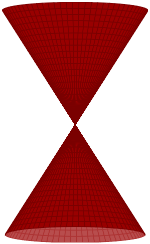

Theorem

The variety V/G is smooth if and only if G is generated by reflections, i.e. $g \in \mathrm{GL}(V)$ with $\mathrm{rk}(g - 1) = 1$.

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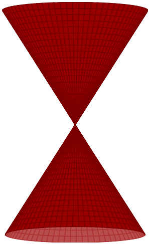

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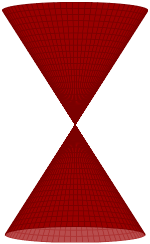

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Corollary

If V is symplectic and $G \leq \mathrm{Sp}(V)$, then V/G is singular.

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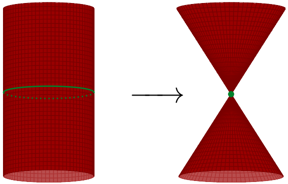

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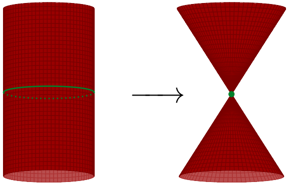

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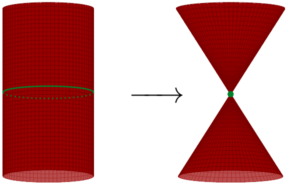

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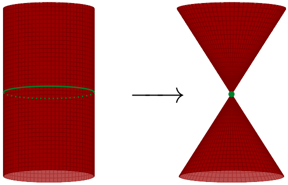

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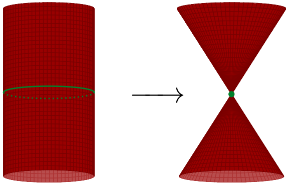

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In general, those do not exist!

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Example

Let $W \leq \mathrm{GL}(\mathfrak{h})$ be a complex reflection group. Then $W^\vee \leq \mathrm{Sp}(\mathfrak{h} \oplus \mathfrak{h}^*)$ is a symplectic reflection group.

Symplectic Reflection Groups

Classification by Cohen, 1980

- Symplectic reflection groups $\dim V \geq 4$
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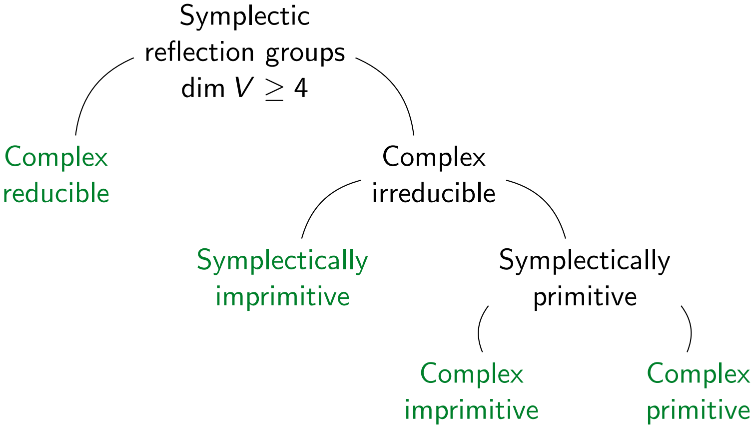

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Using Champ (Thiel, 2013), we arrived at 39 open cases.

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For these groups we have $\dim V = 4$ (Cohen, 1980), so we may assume $V = \mathbb{C}^4$.

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We have the following infinite families of groups in $\mathrm{GL}_2(\mathbb{C})$:

(1) $\mu_d \mathrm{T}$, with d a multiple of 6,

(2) $\mu_d \mathrm{O}$, with d a multiple of 4,

(3) $\mu_d \mathrm{I}$, with d a multiple of 4, 6, or 10,

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The group OT_d is defined as follows. We have $\mathrm{T} \trianglelefteq \mathrm{O}$ with $\mathrm{O}/\mathrm{T} \cong C_2$, so $\mathrm{O} = \langle \mathrm{T}, g \rangle$. We then set

$$\mathrm{OT}_d = \bigcup_{\substack{k=0 \\ k \text{ even}}}^{2d-2} \zeta_{2d}^k \mathrm{T} \cup \bigcup_{\substack{k=1 \\ k \text{ odd}}}^{2d-1} \zeta_{2d}^k g \mathrm{T} \, .$$

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Theorem (Cohen, 1980)

The four infinite families arising in this way give all the symplectically primitive complex imprimitive symplectic reflection groups up to conjugacy.

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μ_4I	22	μ_6I	20
μ_{10}I	16	μ_{12}I	21
μ_{20}I	17	μ_{30}I	18
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Note: This splits the four families in 17 $(2+4+7+4)$ infinite families.
E.g. $\mu_{6d}\mathrm{T}$ for d odd (with $H_0 = G_5$) and $\mu_{6d}\mathrm{T}$ for d even (with $H_0 = G_7$).

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$\mu_{60}\mathrm{I}$	19		
OT_2	12	OT_4	8
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The Classification Problem

The Groups in Question

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Construction of the Module

Outlook

Deformations

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Etingof–Ginzburg, 2002

Such deformations are given by the symplectic reflection algebras $\mathsf{H}_{\mathbf{c}}(V, G)$.

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$$\left\langle [u, v] - \sum_{g \in \mathcal{S}(G)} \mathbf{c}(g)\omega_g(u, v)g \;\middle|\; u, v \in V \right\rangle .$$

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From now on: $\mathsf{H}_{\mathbf{c}}(G) := \mathsf{H}_{\mathbf{c}}(G, V)$.

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If V/G admits a symplectic resolution, then the variety $\operatorname{Spec} Z(\mathsf{H}_{\mathbf{c}}(G))$ is smooth for generic $\mathbf{c}$.

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The same strategy was already used for the complex reducible symplectic reflection groups.

Reducing to Groups I
Rigid Representations

Let M by a G-module. This extends to a module $\tilde{M}$ over $T(V) \rtimes G$ by letting V act trivially.

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Technical detail

The module M is $\mathbf{c}$-rigid if and only if

$$\sum_{g \in \mathcal{S}(G)} \mathbf{c}(g)\omega_g(u,v)\rho_M(g) = 0$$

for all $u, v \in V$, where ρ_M is the representation corresponding to M.

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Example (Thiel, 2014)

Let $D_d \leq \mathrm{GL}_2(\mathbb{C})$ be a dihedral group of order ≥ 10 and consider $D_d^\vee \leq \mathrm{Sp}_4(\mathbb{C})$. Then almost all simple D_d-modules are $\mathbf{c}$-rigid for all $\mathbf{c}$.

Reducing to Groups II
Baby Verma Modules

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For any irreducible W-module λ there exists an irreducible $\mathsf{H}_{\mathbf{c}}(W^\vee)$-module $L(\lambda)$ of dimension $\dim L(\lambda) \leq |W|$.

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One can explicitly compute these modules using an algorithm by Thiel (2015).

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This means we can "split" $\mathbf{c}$ into two parameters $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$ with

$$\mathbf{c}_1 : \mathcal{S}(H_0^\vee) \to \mathbb{C} \text{ and } \mathbf{c}_2 : \mathcal{S}(D_d) \to \mathbb{C}\,.$$

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For $\lambda \in \mathrm{Irr}(H)$, we have $\lambda|_{H_0} \in \mathrm{Irr}(H_0)$ giving rise to a simple $\mathsf{H}_{\mathbf{c}_1}(H_0^\vee)$-module $L(\lambda|_{H_0})$ with $\dim L(\lambda|_{H_0}) \leq |H_0|$.

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Theorem

This is an $\mathsf{H}_{\mathbf{c}}(E(H))$-module if and only if all the constituents of $M|_{D_d}$ are $\mathbf{c}_2$-rigid w.r.t. $\mathsf{H}_{\mathbf{c}_2}(D_d)$.

$$\mathsf{H}_{\mathbf{c}_1}(H_0^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(H^\vee) \subseteq \mathsf{H}_{\mathbf{c}_1}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}}(E(H)) \leftrightsquigarrow \mathsf{H}_{\mathbf{c}_2}(D_d)$$

Conclusion

We can construct an $\mathsf{H}_{\mathbf{c}}(E(H))$-module M with $\dim M < |E(H)|$, if we can find λ such that $L(\lambda|_{H_0})|_{D_d}$ is $\mathbf{c}_2$-rigid and $H_0 \lneq H$.

Conclusion

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Using theoretical bounds on d we obtain:

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Such a module exists except in possibly 73 cases.

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Using the Magma-package Champ (Thiel, 2013) we can improve those theoretical bounds and obtain sharp bounds in many cases:

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Refined Theorem

Such a module exists except in possibly 39 cases. In at least 18 of them no such module exists.

Final result (so far)

Let $G \leq \mathrm{Sp}(V)$ be a symplectically primitive complex imprimitive symplectic reflection group. Then the corresponding quotient V/G does not admit a symplectic resolution except in possibly 39 cases.

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If there is no symplectic resolution, there is still a (singular) $\mathbb{Q}$-factorial terminalization/partial symplectic resolution by the MMP.

- Can we explicitly construct those?
- How many are there? How are they related?
- What is their Cox ring?
- Can we construct the movable cone? $\to$ Connected to a hyperplane arrangement in the parameter space of $\mathsf{H}_{\mathbf{c}}(G)$.

Open questions II

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Theorem (Steinberg, 1964; Bellamy–S.–Thiel, 2021)

A parabolic subgroup of a complex (resp. symplectic) reflection group is as well a complex (resp. symplectic) reflection group.

Open questions II

What properties of complex reflection groups can be translated to symplectic reflection groups?

Theorem (Steinberg, 1964; Bellamy–S.–Thiel, 2021)

A parabolic subgroup of a complex (resp. symplectic) reflection group is as well a complex (resp. symplectic) reflection group.

This is (hopefully) just the beginning:

- There are many equivalent characterizations of complex reflection groups. Can we translate them to symplectic reflection groups? E.g. does $\mathbb{C}[V]^G$ have a "special" structure?
- Translate the notion of reflecting hyperplanes, root systems, etc.
- ...

Open questions III

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Many quotients V/G can be identified with quiver varieties (Bellamy–Craw, 2020; Bellamy–Schedler, 2020; Bellamy et al., 2021).

- For which groups is this the case?
- For any given group G, can we at least find a quiver variety $\mathfrak{M}_0$ and a closed embedding $V/G \hookrightarrow \mathfrak{M}_0$? Is this unique? Or "canonical"?
- If this exists, what is the relationship between the parameter $\mathbf{c}$ of $\mathsf{H}_{\mathbf{c}}(G)$ and the stability parameter ϑ of $\mathfrak{M}_\vartheta$?