Nikolaus School
Computational Geometry

Computing Cox Rings of Linear Quotients in OSCAR

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What we want to compute...

...in theory

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Theorem (Arzhantsev–Gaĭfullin, 2010)

The ring $\mathcal{R}(V/G)$ is $\mathrm{Ab}(G)^\vee$-graded isomorphic to the ring $\mathbb{C}[V]^{[G,G]}$, where the grading is induced by the action of $\mathrm{Ab}(G)$.

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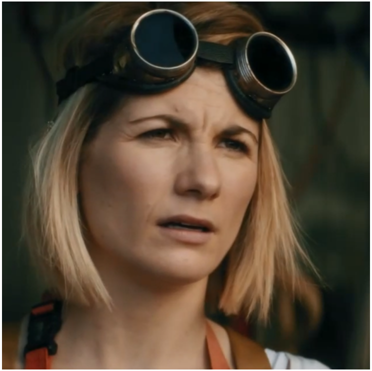
For $\chi \in \mathrm{Ab}(G)^\vee$ we set

$$f \in \mathbb{C}[V]^{[G,G]}_\chi :\Longleftrightarrow \forall \gamma \in \mathrm{Ab}(G) : \gamma.f = \chi(\gamma) f \ .$$

Fascinating!



But what does this mean in practice?

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No problem, calls GAP in the background.

```
julia> K, a = CyclotomicField(3);

julia> r = matrix(K,
              [a 0 0 0; 0 a^-1 0 0; 0 0 a^-1 0; 0 0 0 a]);

julia> s = matrix(K, [0 1 0 0; 1 0 0 0; 0 0 0 1; 0 0 1 0]);

julia> G = matrix_group(r, s)
Matrix group of degree 4 over K

julia> H, HtoG = derived_subgroup(G)
(Matrix group of degree 4 over K, Group homomorphism from
H
to
G)

julia> A, GtoA = maximal_abelian_quotient(GrpAbFinGen, G)
(GrpAb: Z/2, Composite map consisting of the following

G -> Group([ f1 ])
then
Group([ f1 ]) -> A
)
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Since July 2021: Implementation of state-of-the-art algorithms for invariant theory in OSCAR.

```
julia> RH = invariant_ring(H)
Invariant ring of
H
with coefficient ring
K

julia> B, BtoR = affine_algebra(RH);

julia> map(BtoR, gens(B))
12-element Vector{MPolyElem_dec{nf_elem,
                        AbstractAlgebra.Generic.MPoly{nf_elem}}}:
 x[3]*x[4]
 x[2]*x[4]
 x[1]*x[3]
 x[1]*x[2]
 x[4]^3
 x[1]*x[4]^2
 x[1]^2*x[4]
 x[3]^3
 x[2]*x[3]^2
 x[2]^2*x[3]
 x[2]^3
 x[1]^3
```

```
julia> modulus(B)
Ideal generated by -y[9]*y[11] + y[10]^2,
-y[8]*y[11] + y[9]*y[10], -y[3]*y[11] + y[4]*y[10],
-y[1]*y[11] + y[2]*y[10], -y[8]*y[10] + y[9]^2,
-y[3]*y[10] + y[4]*y[9], -y[1]*y[10] + y[2]*y[9],
-y[3]*y[9] + y[4]*y[8], -y[1]*y[9] + y[2]*y[8],
-y[6]*y[12] + y[7]^2, -y[5]*y[12] + y[6]*y[7],
-y[2]*y[12] + y[4]*y[7], -y[1]*y[12] + y[3]*y[7],
-y[5]*y[7] + y[6]^2, -y[2]*y[7] + y[4]*y[6],
-y[1]*y[7] + y[3]*y[6], -y[2]*y[6] + y[4]*y[5],
-y[1]*y[6] + y[3]*y[5], -y[1]*y[4] + y[2]*y[3],
y[4]^3 - y[11]*y[12], y[3]*y[4]^2 - y[10]*y[12],
y[2]*y[4]^2 - y[7]*y[11], y[1]*y[4]^2 - y[7]*y[10],
y[3]^2*y[4] - y[9]*y[12], y[1]*y[3]*y[4] - y[7]*y[9],
y[2]^2*y[4] - y[6]*y[11], y[1]*y[2]*y[4] - y[6]*y[10],
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y[2]^3 - y[5]*y[11], y[1]*y[2]^2 - y[5]*y[10],
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This is linear algebra with respect to the vector spaces $\mathbb{C}[V]_d^{[G,G]}$, $d \in \mathbb{Z}_{\geq 0}$, [Donten-Bury–Keicher, 2016].

```
julia> S, StoR, GtoA = cox_ring_linear_quotient(G);

julia> S
Quotient of
Polynomial Ring in 12 variables t[1], t[2], t[3], t[4],
  ..., t[12] over K graded by Z/2Z via t[1] -> [0],
  t[2] -> [0], t[3] -> [0], t[4] -> [1], ..., t[12] -> [1]

julia> map(StoR, gens(S))
12-element Vector{MPolyElem_dec{nf_elem,
                    AbstractAlgebra.Generic.MPoly{nf_elem}}}:
 x[1]*x[2]
 x[3]*x[4]
 x[1]*x[3] + x[2]*x[4]
 -x[1]*x[3] + x[2]*x[4]
 x[1]^3 + x[2]^3
 x[3]^3 + x[4]^3
 x[2]^2*x[3] + x[1]^2*x[4]
 x[2]*x[3]^2 + x[1]*x[4]^2
 -1//2*x[1]^3 + 1//2*x[2]^3
 x[2]^2*x[3] - x[1]^2*x[4]
 x[2]*x[3]^2 - x[1]*x[4]^2
 -1//2*x[3]^3 + 1//2*x[4]^3
```

```
julia> modulus(S)
Ideal generated by t[1]*t[2] - 1//4*t[3]^2 + 1//4*t[4]^2,
1//2*t[2]*t[5] - t[2]*t[9] - 1//4*t[3]*t[7] + 1//4*t[3]*t[10]
+ 1//4*t[4]*t[7] - 1//4*t[4]*t[10], 1//2*t[2]*t[7]
- 1//2*t[2]*t[10] - 1//4*t[3]*t[8] + 1//4*t[3]*t[11]
+ 1//4*t[4]*t[8] - 1//4*t[4]*t[11], 1//2*t[1]*t[7]
+ 1//2*t[1]*t[10] - 1//4*t[3]*t[5] - 1//2*t[3]*t[9]
+ 1//4*t[4]*t[5] + 1//2*t[4]*t[9], 1//2*t[1]*t[8]
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+ 1//4*t[4]*t[7] + 1//4*t[4]*t[10], 1//2*t[1]*t[6]
- t[1]*t[12] - 1//4*t[3]*t[8] - 1//4*t[3]*t[11] + 1//4*t[4]*t[8]
+ 1//4*t[4]*t[11], 1//2*t[2]*t[8] - 1//2*t[2]*t[11]
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-1//2*t[1]*t[7] + 1//2*t[1]*t[10] + 1//4*t[3]*t[5]
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- 1//4*t[4]*t[7] - 1//4*t[4]*t[10], 1//2*t[2]*t[7]
+ 1//2*t[2]*t[10] - 1//4*t[3]*t[8] - 1//4*t[3]*t[11]
- 1//4*t[4]*t[8] - 1//4*t[4]*t[11], -1//2*t[2]*t[8]
- 1//2*t[2]*t[11] + 1//4*t[3]*t[6] - 1//2*t[3]*t[12]
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