Spring School: Group Actions and
Symplectic Singularities
Université de Lille

On the algorithmic construction of symplectic resolutions of quotient singularities

Johannes Schmitt
Ruhr-Universität Bochum
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Symplectic quotient singularities

Mori dream spaces and Cox rings

An algorithm for $\mathcal{R}(X)$

Namikawa's hyperplanes

Symplectic quotient singularities

V: symplectic $\mathbb{C}$-vector space, $\dim(V) = n < \infty$

$G \leq \mathrm{Sp}(V)$: finite group

Fact: The linear quotient V/G is a symplectic variety (Beauville, 2000).

Let $\varphi : X \to V/G$ be a (projective) symplectic resolution. Might not exist.

Let $\varphi : X \to V/G$ be a $\mathbb{Q}$-factorial terminalization. Exists (BCHM, 2010).

Question: Can we construct $X \to V/G$ algorithmically?

Why? + Some history

New tool to check existence of a symplectic resolution

Special case of 'programming the minimal model programme' (Lazić, 2024)

Donten-Bury–Wiśniewski, 2017: Construction of all symplectic resolutions of V/G with $G = Q_8 \rtimes_{\mathbb{Z}/2\mathbb{Z}} D_8$

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Mori dream spaces and Cox rings

In the following: $\varphi : X \to V/G$ symplectic resolution (or a $\mathbb{Q}$-factorial terminalization)

Proposition (Namikawa, 2015)

The variety X is a relative Mori dream space over V/G.

Equivalently: The Cox ring $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra.

Assume that $\mathrm{Cl}(X)$ is free. The Cox ring of X is the algebra

$$\mathcal{R}(X) = \bigoplus_{[D] \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Example

$\mathcal{R}(\mathbb{P}^n) = \mathbb{C}[x_0, \ldots, x_n]$ with the standard grading.

GIT quotients (algebraic version)

Given a $D \in \mathrm{Div}(X)$, we obtain a positively graded algebra

$$S(D) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O}_X(kD))$$

and the variety $X(D) = \operatorname{Proj} S(D)$.

For some D (ample), we have $X(D) \cong X$.

There are only finitely many symplectic resolutions of V/G up to isomorphism.

All symplectic resolutions V/G arise in this way.

A conceptual algorithm

How to construct X:

(1) Compute $\mathcal{R}(X)$ (without knowing X!)

(2) Find a good $D \in \mathrm{Div}(X)$

(3) Compute $S(D)$

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An embedding

Proposition (Grab, 2019)

There is an injective graded morphism

$$\Theta : \mathcal{R}(X) \to \mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}].$$

Fact: We have $\mathrm{Cl}(V/G) \cong \mathrm{Hom}(G, \mathbb{C}^\times)(= \Delta)$ (Benson, 1993).

Theorem (Arzhantsev–Gaĭfullin, 2010)

There is a graded isomorphism

$$\mathcal{R}(V/G) \cong \mathbb{C}[V]^{[G,G]},$$

where the graded component of $\chi \in \Delta$ is given by

$$\mathbb{C}[V]^{[G,G]}_{\chi} = \{f \in \mathbb{C}[V]^{[G,G]} \mid \gamma.f = \chi(\gamma)f \text{ for all } \gamma \in G\}.$$

Example

Let $G \leq \mathrm{Sp}_4(\mathbb{C})$ acting on $V = \mathbb{C}^4$ be generated by

$$r = \begin{pmatrix} \zeta_3 & . & . & . \\ . & \zeta_3^{-1} & . & . \\ . & . & \zeta_3^{-1} & . \\ . & . & . & \zeta_3 \end{pmatrix} \text{ and } s = \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix}.$$

Then $[G, G] = \langle r \rangle \cong C_3$, so $\mathrm{Cl}(V/G) \cong \Delta \cong \mathrm{Ab}(G) = \mathbb{Z}/2\mathbb{Z}$.

The ring $\mathcal{R}(V/G) \cong \mathbb{C}[V]^{[G,G]}$ is generated by:

In degree χ_0:

$$x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ x_1^3 + x_2^3,\ x_3^3 + x_4^3,\ x_2x_3^2 + x_1x_4^2,\ x_2^2x_3 + x_1^2x_4$$

In degree χ_1:

$$x_1x_3 - x_2x_4,\ x_1^3 - x_2^3,\ x_3^3 - x_4^3,\ x_2x_3^2 - x_1x_4^2,\ x_2^2x_3 - x_1^2x_4$$

What G knows

An element $g \in G$ is a symplectic reflection, if g has exactly two eigenvalues $\neq 1$.

Proposition (S., 2024)

Let $H \leq G$ be generated by the symplectic reflections in G and let m be the number of conjugacy classes of symplectic reflections in G. Then

$$\mathrm{Cl}(X) \cong \mathbb{Z}^m \oplus \mathrm{Hom}(G/H, \mathbb{C}^\times)\,.$$

Theorem 'McKay correspondence' (Ito–Reid, 1996)

Every conjugacy class of symplectic reflections defines a valuation $v : \mathbb{C}[V] \setminus \{0\} \to \mathbb{Z}$. These valuations correspond one-to-one to divisorial valuations on X.

Example

The group $G = \langle r, s \rangle$ is a symplectic reflection group, generated by s and rs.

$\Longrightarrow H = G$ and $\mathrm{Hom}(G/H, \mathbb{C}^\times)$ is trivial.

There is one conjugacy class of symplectic reflections (for example, $r(rs)r^{-1} = s$).

$\Longrightarrow m = 1$ and $\mathrm{Cl}(X) \cong \mathbb{Z}$.

For the corresponding valuation v, we have for example $v(x_1 + x_2) = 0$ and $v(x_1 - x_2) = 1$.

Reduction to $\mathcal{R}(V/G)$

There is an injective graded morphism

$$\Theta : \mathcal{R}(X) \to \mathcal{R}(V/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}].$$

Let $\mathrm{Cl}(X)^{\mathrm{free}} \cong \mathbb{Z}^m$, so $\mathbb{C}[\mathrm{Cl}(X)^{\mathrm{free}}] = \mathbb{C}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and we have the valuations $v_1, \dots, v_m$.

Given $f \in \mathcal{R}(V/G)$, we have

$$f \otimes \prod_{i=1}^{m} t_i^{v_i(f)} \in \mathrm{im}(\Theta).$$

Theorem (Yamagishi, 2018; Grab, 2019; S., 2024+)

Homogeneous elements $f_1, \dots, f_k \in \mathcal{R}(V/G)$ give rise to generators of $\mathrm{im}(\Theta)$ if and only if $\{f_1, \dots, f_k\}$ is a Δ-homogeneous Khovanskii basis (or MUVAK basis) of $\mathcal{R}(V/G)$ with respect to $v_1, \dots, v_m$.

Homogeneous Khovanskii bases

Idea: Special generators whose 'initial terms' generate the 'initial algebra'.

$m = 1$: Khovanskii basis
$m > 1$: MUVAK basis

A finite Khovanskii basis may not exist in a more general setting, but here it does ($\rightarrow$ MDS).

There is an algorithm.

Example

In the example, the given generators of $\mathbb{C}[V]^{[G,G]}$ already form a homogeneous Khovanskii basis.

Generators of $\mathcal{R}(X) \subseteq \mathbb{C}[V] \otimes \mathbb{C}[t^{\pm 1}]$ are given by

$$x_1x_2,\ x_3x_4,\ x_1x_3 + x_2x_4,\ x_1^3 + x_2^3,\ x_3^3 + x_4^3,\ x_2x_3^2 + x_1x_4^2,\ x_2^2x_3 + x_1^2x_4$$
$$(x_1x_3 - x_2x_4)t,\ (x_1^3 - x_2^3)t,\ (x_3^3 - x_4^3)t,\ (x_2x_3^2 - x_1x_4^2)t,\ (x_2^2x_3 - x_1^2x_4)t, t^{-2}$$

where the grading by $\mathrm{Cl}(X) = \mathbb{Z}$ is via the degree of t.

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Namikawa's hyperplanes

There is a wall-and-chamber structure in a subcone of $\mathbb{R}^m$.

Chambers $\hat{=}$ isomorphism classes of $\mathbb{Q}$-factorial terminalizations.

Walls give divisors D such that $X(D)$ is not a $\mathbb{Q}$-factorial terminalization.

Namikawa, 2015: The walls come from a hyperplane arrangement and there is a reflection group acting on this arrangement.

This requires that V/G is symplectic.