Nikolaus Conference 2025

Invariants in the cohomology of the complement of quaternionic reflection arrangements

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Joint with
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- $V = \mathbb{H}^n$ and $K \wr S_n$ with $K \leq \mathbb{H}^\times$ finite

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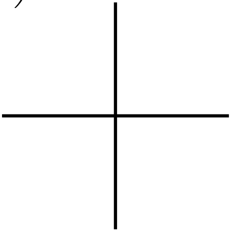

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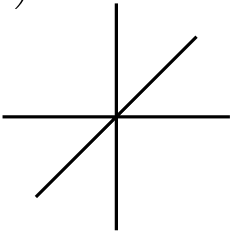

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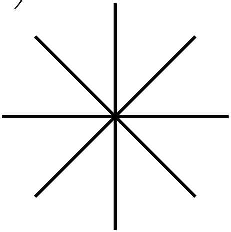

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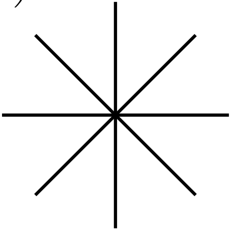

The reflection arrangement of G is the set

$$\mathcal{A}_G = \{\mathrm{Fix}(g) \mid g \in G \text{ reflection}\}.$$

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- Cadegan-Schlieper (2018): "Orlik–Solomon presentation"

$$H^*(M(\mathcal{A}_G);\mathbb{Q}) \cong \Lambda(\mathbb{Q}^m)/I$$

with $m = |\mathcal{A}_G|$ and I comes from dependency relations in $\mathcal{A}_G$

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For complex reflection groups: Lehrer (2004), Callegaro–Marin (2014), Marin (2017), and Douglass–Pfeiffer–Röhrle (2025)

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Lemma

If $\dim(V) = 2$ and G acts on $\mathcal{A}_G$ with k orbits, then

$$P(\mathcal{A}_G, G; t) = 1 + kt^3 + (k-1)t^6.$$

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Proposition

We have

$$H^{3k}(M(\mathcal{A}_G);\mathbb{Q})^G \cong \bigoplus_{P \in X_k} H^{3k}(M(\mathcal{A}_P);\mathbb{Q})^{N_G(P)}$$

as $\mathbb{Q}G$-modules.

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- $H \neq \{1\} \Longrightarrow$ the lattice of parabolic subgroups is the Dowling lattice of K

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Theorem

Let $G = G_n(K, H)$ with K not cyclic. Then we have

$$P(\mathcal{A}_G, G; t) = 1 + 2t^3 + \cdots + 2t^{3n-6} + at^{3n-3} + (a-1)t^{3n}$$

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with

$$a = \begin{cases} 3, & [K : H] \text{ and } n \text{ are even and } K/H \text{ is cyclic,} \\ 5, & [K : H] \text{ and } n \text{ are even and } K/H \text{ is not cyclic,} \\ 2, & \text{otherwise.} \end{cases}$$

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G	n	$P(\mathcal{A}_G, G; t)$
$W(Q)$	3	$1 + t^3$
$W(R)$	3	$1 + t^3$
$W(S_1)$	4	$1 + t^3 + t^9 + t^{12}$
$W(S_2)$	4	$1 + t^3 + t^9 + t^{12}$
$W(S_3)$	4	$1 + t^3 + t^9 + t^{12}$
$W(T)$	4	$1 + t^3 + t^9 + t^{12}$
$W(U)$	5	$1 + t^3 + t^{12} + t^{15}$